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Non-diagonal solutions of the reflection equation for the trigonometric $A_{n-1}^{(1)}$ vertex model

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Abstract: We obtain a class of non-diagonal solutions of the reflection equation for the trigonometric $A_{n-1}^{(1)}$ vertex model. The solutions can be expressed in terms of intertwinner matrix and its inverse, which intertwine two trigonometric R-matrices. In addition to a *discrete* (positive integer) parameter l, $1 \leq l \leq n$, the solution contains $n+2$ *continuous* boundary parameters.

Keywords: Quantum Groups, Lattice Integrable Models, Boundary Quantum Field Theory.

Contents

1. Introduction

Two-dimensional integrable models have traditionally been solved by imposing periodic boundary conditions. For such bulk systems, the quantum Yang-Baxter equation (QYBE)

$$R_{12}(u_1-u_2)R_{13}(u_1-u_3)R_{23}(u_2-u_3)=R_{23}(u_2-u_3)R_{13}(u_1-u_3)R_{12}(u_1-u_2)\,, \quad (1.1)$$

leads to families of commuting row-to-row *transfer matrices* which may be diagonalized by the quantum inverse scattering method (QISM) [1]. The QYBE has been studied for a long time and a large number of solutions, the so-called R-matrices, are known.

Not all boundary conditions are compatible with integrability in the bulk. The bulk integrability only holds when one imposes integrable boundary conditions specified by the so-called boundary K-matrices $K^{\pm}(u)$ which satisfy the reflection equation (RE) and its dual [2]–[4]. The RE (also called the boundary Yang-Baxter equation) and its dual are the analogue of the QYBE for models with open boundaries.

Much effort has been made on constructing (*non-diagonal*) solutions (K-matrices) of the RE by brute force (i.e. solving the RE directly) [4]–[10] and more recently by the boundary quantum group approach [11]–[13]. In this paper, we propose a different approach, which is referred to as the intertwiner-matrix approach, to construct a class of non-diagonal solutions of the RE associated with the trigonometric $A_{n-1}^{(1)}$ vertex model. The idea is originated from the studies [9, 10] of the RE associated with the $\mathcal{Z}_n$ elliptic Belavin model [14]. It has been observed in [15] that the generic non-diagonal solutions [4, 5] of the RE associated with the trigonometric $A_1^{(1)}$ vertex model (spin-$\frac{1}{2}$ XXZ model), as in the $\mathcal{Z}_n$ Belavin model, can be expressed in terms of the intertwiner-matrices and the diagonal face-type K-matrix. The advantage of this face-vertex correspondence method is that in the "face" picture the corresponding K-matrices become diagonal. This enables one to diagonalize the corresponding double-row transfer matrix with the special choice of the left and right boundary parameters [16] by the generalized QISM developed in [17]. Despite the success of this approach in the trigonometric $A_1^{(1)}$ and $\mathcal{Z}_n$ elliptic vertex boundary models, to our knowledge the face-vertex correspondence for boundary trigonometric $A_{n-1}^{(1)}$ model with $n > 2$ has not been achieved so far. This is also in contrast to the boundary rational $A_{n-1}^{(1)}$ model where the non-diagonal K-matrices are equivalent to the diagonal ones by simple spectral-independent similarity transformations [18, 19].

We have organized the paper as follows. In section 2, we introduce our notation and some basic ingredients. In section 3, we construct the intertwiner-matrix which satisfies the face-vertex correspondence relation between the two R-matrices $R(u)$ and $W(u)$. Through the *magic* intertwiner-matrix, we obtain, in section 4, a set of non-diagonal solutions to the RE and the dual RE. Section 5 is devoted to the conclusion. A sketch of the proof of the face-vertex correspondence relation is provided in the appendix.

2. $A_{n-1}^{(1)}$ reflection equation

Let us fix a positive integer n ($n \geq 2$) and a generic complex number η, and $R(u) \in End(\mathcal{C}^n \otimes \mathcal{C}^n)$ be the trigonometric solution to the $A_{n-1}^{(1)}$ type QYBE given by [20, 21, 22]

$$R(u) = \sum_{\alpha=1}^{n} R^{\alpha\alpha}_{\alpha\alpha}(u) E_{\alpha\alpha} \otimes E_{\alpha\alpha} + \sum_{\alpha\neq\beta} \left\{ R^{\alpha\beta}_{\alpha\beta}(u) E_{\alpha\alpha} \otimes E_{\beta\beta} + R^{\beta\alpha}_{\alpha\beta}(u) E_{\beta\alpha} \otimes E_{\alpha\beta} \right\}, \quad (2.1)$$

where E_{ij} is the matrix with elements $(E_{ij})^l_k = \delta_{jk}\delta_{il}$. The coefficient functions are

$$R^{\alpha\beta}_{\alpha\beta}(u) = \begin{cases} \frac{\sin(u)\, e^{-i\eta}}{\sin(u+\eta)}, & \alpha > \beta, \\ \quad 1, & \alpha = \beta, \\ \frac{\sin(u)\, e^{i\eta}}{\sin(u+\eta)}, & \alpha < \beta, \end{cases} , \quad (2.2)$$

$$R^{\beta\alpha}_{\alpha\beta}(u) = \begin{cases} \frac{\sin(\eta)\, e^{iu}}{\sin(u+\eta)}, & \alpha > \beta, \\ \quad 1, & \alpha = \beta, \\ \frac{\sin(\eta)\, e^{-iu}}{\sin(u+\eta)}, & \alpha < \beta, \end{cases} . \quad (2.3)$$

One can check that the R-matrix satisfies the following unitarity, crossing-unitarity and quasi-classical relations:

$$\text{Unitarity}: \quad R_{12}(u)R_{21}(-u) = \text{id}, \quad (2.4)$$

$$\text{Crossing-unitarity}: \quad R^{t_2}_{12}(u)M_2^{-1}R^{t_2}_{21}(-u-n\eta)M_2 = \frac{\sin(u)\sin(u+n\eta)}{\sin(u+\eta)\sin(u+n\eta-\eta)}\,\text{id}, \quad (2.5)$$

$$\text{Quasi-classical property}: \quad R_{12}(u)|_{\eta\to 0} = \text{id}. \quad (2.6)$$

Here $R_{21}(u) = P_{12}R_{12}(u)P_{12}$ with P_{12} being the usual permutation operator and t_i denotes the transposition in the i-th space, and η is the so-called crossing paramter. The crossing matrix M is a diagonal $n \times n$ matrix with elements

$$M_{\alpha\beta} = M_\alpha \delta_{\alpha\beta}\,, \qquad M_\alpha = e^{-2i\alpha\eta}\,, \qquad \alpha = 1, \ldots, n\,. \tag{2.7}$$

Here and below we adopt the standard notation: for any matrix $A \in \mathrm{End}(\mathcal{C}^n)$, A_j is an embedding operator in the tensor space $\mathcal{C}^n \otimes \mathcal{C}^n \otimes \cdots$, which acts as A on the j-th space and as an identity on the other factor spaces; $R_{ij}(u)$ is an embedding operator of R-matrix in the tensor space, which acts as an identity on the factor spaces except for the i-th and j-th ones.

One introduces the "row-to-row" monodromy matrix $T(u)$, which is an $n \times n$ matrix with elements being operators acting on $(\mathcal{C}^n)^{\otimes N}$

$$T(u) = R_{01}(u + z_1) R_{02}(u + z_2) \cdots R_{0N}(u + z_N)\,. \tag{2.8}$$

Here $\{z_i | i = 1, \ldots, N\}$ are arbitrary free complex parameters which are usually called inhomogeneous parameters. With the help of the QYBE (1.1), one can show that $T(u)$ satisfies the so-called "RLL" relation

$$R_{12}(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u - v)\,. \tag{2.9}$$

Integrable open chains can be constructed as follows [2]. Let us introduce the K-matrix $K^-(u)$ which gives rise to an integrable boundary condition on the right boundary. $K^-(u)$ satisfies the RE

$$R_{12}(u_1 - u_2) K_1^-(u_1) R_{21}(u_1 + u_2) K_2^-(u_2) = K_2^-(u_2) R_{12}(u_1 + u_2) K_1^-(u_1) R_{21}(u_1 - u_2)\,. \tag{2.10}$$

For models with open boundaries, instead of the standard "row-to-row" monodromy matrix $T(u)$ (2.8), one needs the "double-row" monodromy matrix $\mathcal{T}(u)$

$$\mathcal{T}(u) = T(u) K^-(u) T^{-1}(-u)\,. \tag{2.11}$$

Using (2.9) and (2.10), one can prove that $\mathcal{T}(u)$ satisfies

$$R_{12}(u_1 - u_2) \mathcal{T}_1(u_1) R_{21}(u_1 + u_2) \mathcal{T}_2(u_2) = \mathcal{T}_2(u_2) R_{12}(u_1 + u_2) \mathcal{T}_1(u_1) R_{21}(u_1 - u_2)\,. \tag{2.12}$$

In order to construct the *double-row transfer matrices*, besides the RE, one needs another K-matrix $K^+(u)$ which gives integrable boundary condition on the left boundary. The explicit form of the dual RE is related with the crossing-unitarity relation of the R-matrix [2, 3]. For a R-matrix whose partial transpositions have inverse, the corresponding dual RE reads

$$\begin{aligned} &R_{12}(u_2 - u_1) K_1^+(u_1) R_{21}^{-1,t_2,-1,t_2}(-u_1 - u_2) K_2^+(u_2) = \\ &\quad = K_2^+(u_2) R_{12}^{-1,t_1,-1,t_1}(-u_1 - u_2) K_1^+(u_1) R_{21}(u_2 - u_1)\,. \end{aligned} \tag{2.13}$$

The crossing-unitarity relation (2.5) of the R-matrix $R(u)$ allows one to simplify the dual RE into the form

$$\begin{aligned} &R_{12}(u_2 - u_1) K_1^+(u_1)\, M_1^{-1}\, R_{21}(-u_1 - u_2 - n\eta)\, M_1\, K_2^+(u_2) = \\ &\quad = M_1\, K_2^+(u_2) R_{12}(-u_1 - u_2 - n\eta)\, M_1^{-1}\, K_1^+(u_1) R_{21}(u_2 - u_1)\,. \end{aligned} \tag{2.14}$$

JHEP12(2004)019

Different integrable boundary conditions are described by different solutions $K^-(u)$ ($K^+(u)$) to the (dual) RE [2, 4]. Then the *double-row transfer matrix* of the inhomogeneous model associated with the R-matrix (2.1)–(2.3) with open boundary specified by the K-matrices $K^{\pm}(u)$ is given by

$$\tau(u) = tr(K^+(u)\mathcal{T}(u))\,. \tag{2.15}$$

The commutativity of the transfer matrices

$$[\tau(u), \tau(v)] = 0\,, \tag{2.16}$$

follows as a consequence of (1.1), (2.4)–(2.5) and (2.12)–(2.13). This ensures the integrability of the inhomogeneous model with open boundary. In this paper, we search for non-diagonal solutions to the RE (2.10) with multiple continuous parameters (c.f. [5]) by means of the intertwiner-matrix approach.

3. Intertwining vectors and the associated face-vertex correspondence relations

Let $\{\epsilon_i \mid i = 1, 2, \ldots, n\}$ be the orthonormal basis of the vector space $\mathcal{C}^n$ such that $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$. For a vector $m \in \mathcal{C}^n$, define

$$m_i = \langle m, \epsilon_i \rangle\,, \qquad |m| = \sum_{l=1}^{n} m_l\,, \qquad i = 1, \ldots, n\,. \tag{3.1}$$

Let us introduce n intertwiners $\{\phi_{m,m-\epsilon_j}(u) | \, j = 1, \ldots, n\}$. Each $\phi_{m,m-\epsilon_j}(u)$ is an n-component column vector whose α-th elements are $\{\phi^{(\alpha)}_{m,m-\epsilon_j}(u)\}$. The n intertwiners form an $n \times n$ matrix (in which j and α stand for the column and the row indices respectively), called the intertwiner-matrix, with the non-vanishing matrix elements being

$$\begin{pmatrix} e^{i\eta f_1(m)} & & & & & & & e^{i\eta F_n(m)+\rho_n} e^{2iu} \\ e^{i\eta F_1(m)+\rho_1} & e^{i\eta f_2(m)} & & & & & & \\ & e^{i\eta F_2(m)+\rho_2} & \ddots & & & & & \\ & & \ddots & e^{i\eta f_j(m)} & & & & \\ & & & e^{i\eta F_j(m)+\rho_j} & \ddots & & & \\ & & & & \ddots & e^{i\eta f_{n-1}(m)} & & \\ & & & & & e^{i\eta F_{n-1}(m)+\rho_{n-1}} & & e^{i\eta f_n(m)} \end{pmatrix}. \tag{3.2}$$

Here $\{\rho_i | i = 1, \ldots, n\}$ are complex constants with regard to u and m, and $\{f_i(m) | i = 1, \ldots, n\}$ and $\{F_i(m) | i = 1, \ldots, n\}$ are linear functions of m:

$$f_i(m) = \sum_{l=1}^{i-1} m_l - m_i - \frac{1}{2}|m|\,, \qquad i = 1, \ldots, n\,, \tag{3.3}$$

$$F_i(m) = \sum_{l=1}^{i} m_l - \frac{1}{2}|m|\,, \qquad i = 1, \ldots, n-1\,, \tag{3.4}$$

$$F_n(m) = -\frac{3}{2}|m|\,. \tag{3.5}$$

By the definitions of $f_j(m)$ and $F_j(m)$, after a straightforward calculation, we find

$$f_j(\epsilon_k) = \begin{cases} \frac{1}{2}, & j > k, \\ -\frac{3}{2}, & j = k, \\ -\frac{1}{2}, & j < k, \end{cases} \tag{3.6}$$

$$F_j(\epsilon_k) = \begin{cases} \frac{1}{2}, & j \geq k, \\ -\frac{1}{2}, & j < k\,, \end{cases} \qquad \text{for } j \neq n\,, \tag{3.7}$$

$$F_n(\epsilon_k) = -\frac{3}{2}\,. \tag{3.8}$$

The above equations allow us to derive the following *face-vertex correspondence relation*:

$$\begin{aligned} R_{12}(u_1 - u_2)\phi_{m,m-\epsilon_i}(u_1) \otimes \phi_{m-\epsilon_i,m-\epsilon_i-\epsilon_j}(u_2) = \\ = \sum_{kl} W_{ij}^{kl}(u_1 - u_2)\phi_{m-\epsilon_l,m-\epsilon_l-\epsilon_k}(u_1) \otimes \phi_{m,m-\epsilon_l}(u_2)\,. \end{aligned} \tag{3.9}$$

Here the non-vanishing elements of $\{W(u)_{ij}^{kl}\}$ are

$$W_{jj}^{jj}(u) = 1\,, \qquad W_{jk}^{jk}(u) = \frac{\sin(u)}{\sin(u+\eta)}, \quad \text{for } j \neq k\,, \tag{3.10}$$

$$W_{jk}^{kj}(u) = \begin{cases} \frac{\sin(\eta)\,e^{iu}}{\sin(u+\eta)}\,, & j > k\,, \\ \frac{\sin(\eta)\,e^{-iu}}{\sin(u+\eta)}\,, & j < k\,, \end{cases} \qquad \text{for } j \neq k\,. \tag{3.11}$$

A sketch of proof of the above face-vertex correspondence relation is relegated to the appendix. Then associated with $\{W(u)_{ij}^{kl}\}$, one can introduce "*face type*" R-matrix $W(u)$

$$W(u) = \sum_{i,j,k,l} W_{ij}^{kl}(u) E_{ki} \otimes E_{lj}\,. \tag{3.12}$$

Some remarks are in order. The face type R-matrix $W(u)$ does not depend on the face type parameter m, in contrast to the $\mathcal{Z}_n$ elliptic case [14, 23]. It follows that $W(u)$ and $R(u)$ satisfy the same QYBE, i.e. $W(u)$ obeys the usual (vertex type) QYBE rather than the *dynamical* one [24, 17].

The forms of the R-matrices $R(u)$ (2.1)–(2.3) and $W(u)$ (3.10)–(3.12) ensure that the set of intertwiner matrices satisfying the face-vertex correspondence relation (3.9) is invariant under the similarity transformation given by an arbitrary *diagonal constant* matrix. This allows one to turn the intertwiner-matrix (3.2) into the following *standard* form:

$$\begin{pmatrix} e^{i\eta f_1(m)} & & & & & & e^{i\eta F_n(m)+\rho}e^{2iu} \\ e^{i\eta F_1(m)} & e^{i\eta f_2(m)} & & & & & \\ & e^{i\eta F_2(m)} & \ddots & & & & \\ & & \ddots & e^{i\eta f_j(m)} & & & \\ & & & e^{i\eta F_j(m)} & \ddots & & \\ & & & & \ddots & e^{i\eta f_{n-1}(m)} & \\ & & & & & e^{i\eta F_{n-1}(m)} & e^{i\eta f_n(m)} \end{pmatrix}, \tag{3.13}$$

where ρ is related to the original parameters $\{\rho_i\}$ in (3.2) by $\rho = \sum_{l=1}^{n} \rho_l$, and the linear functions $\{f_j(m)\}$ and $\{F_j(m)\}$ are the same as those given by (3.3)–(3.5). Noting that

$$\sum_{i=1}^{n} f_i(m) = \sum_{i=1}^{n} F_i(m) = \sum_{l=1}^{n} \frac{n-2(l+1)}{2} m_l\,, \tag{3.14}$$

one can show that the determinant of the intertwiner matrix (3.13) is

$$\mathrm{Det}\left(\phi^{(\alpha)}_{m,m-\epsilon_j}(u)\right) = e^{i\eta \sum_{l=1}^{n} \frac{n-2(l+1)}{2} m_l} \left(1-(-1)^n e^{2iu+\rho}\right). \tag{3.15}$$

For a generic $\rho \in \mathcal{C}$ this determinant is not vanishing and thus the inverse of (3.13) exists. This fact allows us to introduce other types of intertwiners $\bar{\phi}$ and $\tilde{\phi}$ satisfying the following orthogonality conditions:

$$\sum_{\alpha} \bar{\phi}^{(\alpha)}_{m,m-\epsilon_i}(u)\phi^{(\alpha)}_{m,m-\epsilon_j}(u) = \delta_{ij}\,, \tag{3.16}$$

$$\sum_{\alpha} \tilde{\phi}^{(\alpha)}_{m+\epsilon_i,m}(u)\phi^{(\alpha)}_{m+\epsilon_j,m}(u) = \delta_{ij}\,. \tag{3.17}$$

From these conditions we derive the "completeness" relations:

$$\sum_{k} \bar{\phi}^{(\alpha)}_{m,m-\epsilon_k}(u)\phi^{(\beta)}_{m,m-\epsilon_k}(u) = \delta_{\alpha\beta}\,, \tag{3.18}$$

$$\sum_{k} \tilde{\phi}^{(\alpha)}_{m+\epsilon_k,m}(u)\phi^{(\beta)}_{m+\epsilon_k,m}(u) = \delta_{\alpha\beta}\,. \tag{3.19}$$

With the help of (3.16)–(3.19), we obtain, from the face-vertex correspondence relation (3.9),

$$\begin{aligned}&\left(\tilde{\phi}_{m+\epsilon_k,m}(u_1)\otimes \mathrm{id}\right) R_{12}(u_1-u_2)\left(\mathrm{id}\otimes \phi_{m+\epsilon_j,m}(u_2)\right) = \\ &\quad= \sum_{i,l} W^{kl}_{ij}(u_1-u_2)\,\tilde{\phi}_{m+\epsilon_i+\epsilon_j,m+\epsilon_j}(u_1)\otimes \phi_{m+\epsilon_k+\epsilon_l,m+\epsilon_k}(u_2)\,,\end{aligned} \tag{3.20}$$

$$\begin{aligned}&\left(\tilde{\phi}_{m+\epsilon_k,m}(u_1)\otimes \tilde{\phi}_{m+\epsilon_k+\epsilon_l,m+\epsilon_k}(u_2)\right) R_{12}(u_1-u_2) = \\ &\quad= \sum_{i,j} W^{kl}_{ij}(u_1-u_2)\,\tilde{\phi}_{m+\epsilon_i+\epsilon_j,m+\epsilon_j}(u_1)\otimes \tilde{\phi}_{m+\epsilon_j,m}(u_2)\,,\end{aligned} \tag{3.21}$$

$$\begin{aligned}&\left(\mathrm{id}\otimes \bar{\phi}_{m,m-\epsilon_l}(u_2)\right) R_{12}(u_1-u_2)\left(\phi_{m,m-\epsilon_i}(u_1)\otimes \mathrm{id}\right) = \\ &\quad= \sum_{k,j} W^{kl}_{ij}(u_1-u_2)\,\phi_{m-\epsilon_l,m-\epsilon_k-\epsilon_l}(u_1)\otimes \bar{\phi}_{m-\epsilon_i,m-\epsilon_i-\epsilon_j}(u_2)\,,\end{aligned} \tag{3.22}$$

$$\begin{aligned}&\left(\bar{\phi}_{m-\epsilon_l,m-\epsilon_k-\epsilon_l}(u_1)\otimes \bar{\phi}_{m,m-\epsilon_l}(u_2)\right) R_{12}(u_1-u_2) = \\ &\quad= \sum_{i,j} W^{kl}_{ij}(u_1-u_2)\,\bar{\phi}_{m,m-\epsilon_i}(u_1)\otimes \bar{\phi}_{m-\epsilon_i,m-\epsilon_i-\epsilon_j}(u_2)\,.\end{aligned} \tag{3.23}$$

4. Non-diagonal solutions to the reflection equation

We first present the main new result of this paper. It can be shown (see subsection 4.1) that

$$K^{-}(u)^s_t = \sum_{i=1}^{n} k_i(u)\phi^{(s)}_{\lambda,\lambda-\epsilon_i}(u)\bar{\phi}^{(t)}_{\lambda,\lambda-\epsilon_i}(-u)\,, \tag{4.1}$$

gives a class of non-diagonal solutions of the RE associated with the trigonometric R-matrix (2.1)–(2.3). Here $\{k_i(u)|i=1,\ldots,n\}$ are the matrix elements of the λ-independent face-type diagonal matrix $\mathcal{K}(\lambda|u)$,

$$\mathcal{K}(\lambda|u)=\mathrm{Diag}\left(k_1(u),\ldots,k_n(u)\right)\,, \tag{4.2}$$

where

$$k_j(u)=\begin{cases}1, & 1\leq j\leq l,\\ \frac{\sin(\xi-u)}{\sin(\xi+u)}e^{-2iu}, & l+1\leq j\leq n.\end{cases} \tag{4.3}$$

$\phi_{\lambda,\lambda-\epsilon_i}(u)$ and $\bar{\phi}_{\lambda,\lambda-\epsilon_i}(u)$ are respectively given by the i-th column of (3.13) and (3.16) specializing to $m=\lambda=\sum_{i=1}^n\lambda_i\epsilon_i$, and l is an positive integer, $1\leq l\leq n$. In addition to the discrete boundary parameter l (c.f. [5]), the solution contains $n+2$ *continuous* boundary parameters ξ, ρ and $\{\lambda_i|i=1,\ldots,n\}$. Besides the RE (2.10) (which will be proved later), the K-matrix $K^-(u)$ (4.1) also satisfies the *regular* condition (c.f. [7, 12]):

$$K^-(0)=\mathrm{id}\,, \tag{4.4}$$

and boundary unitarity relation:

$$K^-(u)\,K^-(-u)=\mathrm{id}\,. \tag{4.5}$$

Eqs. (4.4) and (4.5) follow from the orthogonality condition (3.16) and the "completeness" relation (3.18) specializing to $m=\lambda$. In the following, we shall show that the non-diagonal K-matrix $K^-(u)$ (4.1) satisfies the RE (2.10) with the R-matrix given by (2.1)–(2.3).

4.1 Proof of the RE

Let $K^-(u)$ be a solution to the RE (2.10). As in [10], we introduce the corresponding face-type K-matrix $\mathcal{K}(m|u)$ via

$$\mathcal{K}(m|u)_i^j=\sum_{s,t}\tilde{\phi}^{(s)}_{m-\epsilon_i+\epsilon_j,\ \lambda-\epsilon_i}(u)K^-(u)^s_t\phi^{(t)}_{m,\ m-\epsilon_i}(-u)\,. \tag{4.6}$$

Multiplying both sides of the RE (2.10) from the right by

$$\phi_{m+\epsilon_{i_3},m}(-u_1)\otimes\phi_{m+\epsilon_{i_3}+\epsilon_{j_3},m+\epsilon_{i_3}}(-u_2)\,,$$

and using the face-vertex correspondence relation (3.9) and the "completeness" relation (3.19), we have, for the l.h.s. of the resulting relation,

$$\begin{aligned}\text{l.h.s.}&=R_{12}(u_1-u_2)K_1^-(u_1)R_{21}(u_1+u_2)\times\\ &\quad\times(\phi_{m+\epsilon_{i_3},m}(-u_1)\otimes K^-(u_2)\,\phi_{m+\epsilon_{i_3}+\epsilon_{j_3},m+\epsilon_{i_3}}(-u_2))\\ &=R_{12}(u_1-u_2)K_1^-(u_1)R_{21}(u_1+u_2)(\phi_{m+\epsilon_{i_3},m}(-u_1)\otimes 1)\times\\ &\quad\times\Bigg(1\otimes\Bigg\{\sum_{j_2}\phi_{m+\epsilon_{i_3}+\epsilon_{j_2},m+\epsilon_{i_3}}(u_2)\tilde{\phi}_{m+\epsilon_{i_3}+\epsilon_{j_2},m+\epsilon_{i_3}}(u_2)\times\end{aligned}$$

$$\times K^-(u_2)\phi_{m+\epsilon_{i_3}+\epsilon_{j_3},m+\epsilon_{i_3}}(-u_2)\Big\}\Big)$$

$$= \sum_{j_2} R_{12}(u_1-u_2)K_1^-(u_1)R_{21}(u_1+u_2)\times$$

$$\times\left(\phi_{m+\epsilon_{i_3},m}(-u_1)\otimes\phi_{m+\epsilon_{i_3}+\epsilon_{j_2},m+\epsilon_{i_3}}(u_2)\right)\mathcal{K}(m+\epsilon_{i_3}+\epsilon_{j_3}|u_2)^{j_2}_{j_3}$$

$$= \sum_{i_2}\sum_{j_1,j_2} R_{12}(u_1-u_2)K_1^-(u_1)W^{j_1\,i_2}_{j_2\,i_3}(u_1+u_2)\times$$

$$\times\left(\phi_{m+\epsilon_{i_3}+\epsilon_{j_2},m+\epsilon_{j_1}}(-u_1)\otimes\phi_{m+\epsilon_{j_1},m}(u_2)\right)\mathcal{K}(m+\epsilon_{i_3}+\epsilon_{j_3}|u_2)^{j_2}_{j_3}$$

$$\vdots$$

$$= \sum_{i_0,j_0}(\phi_{m+\epsilon_{i_0},m}(u_1)\otimes\phi_{m+\epsilon_{i_0}+\epsilon_{j_0},m+\epsilon_{i_0}}(u_2))\times$$

$$\times\Bigg\{\sum_{i_1,i_2}\sum_{j_1,j_2}W^{i_0\,j_0}_{i_1\,j_1}(u_1-u_2)\mathcal{K}(m+\epsilon_{i_2}+\epsilon_{j_1}|u_1)^{i_1}_{i_2}\times$$

$$\times W^{j_1\,i_2}_{j_2\,i_3}(u_1+u_2)\mathcal{K}(m+\epsilon_{i_3}+\epsilon_{j_3}|u_2)^{j_2}_{j_3}\Bigg\}. \qquad (4.7)$$

Similarly for the r.h.s. of the resulting relation, we obtain

$$\text{r.h.s.} = \sum_{i_0,j_0}\left(\phi_{m+\epsilon_{i_0},m}(u_1)\otimes\phi_{m+\epsilon_{i_0}+\epsilon_{j_0},m+\epsilon_{i_0}}(u_2)\right)\times$$

$$\times\Bigg\{\sum_{i_1,i_2}\sum_{j_1,j_2}\mathcal{K}(\lambda+\epsilon_{i_0}+\epsilon_{j_1}|u_2)^{j_0}_{j_1}W^{i_0\,j_1}_{i_1\,j_2}(u_1+u_2)\times$$

$$\times\mathcal{K}(m+\epsilon_{i_2}+\epsilon_{j_2}|u_1)^{i_1}_{i_2}W^{j_2\,i_2}_{j_3\,i_3}(u_1-u_2)\Bigg\}. \qquad (4.8)$$

Note that intertwiners are linearly independent, which follows from (3.15). Therefore the K-matrix $K^-(u)$ would satisfy the RE (2.10) provided that the corresponding face-type K-matrix $\mathcal{K}(m|u)$ defined by (4.6) obeys the relation

$$\sum_{i_1,i_2}\sum_{j_1,j_2}W^{i_0\,j_0}_{i_1\,j_1}(u_1-u_2)\mathcal{K}(m+\epsilon_{j_1}+\epsilon_{i_2}|u_1)^{i_1}_{i_2}W^{j_1\,i_2}_{j_2\,i_3}(u_1+u_2)\mathcal{K}(m+\epsilon_{j_3}+\epsilon_{i_3}|u_2)^{j_2}_{j_3} =$$

$$= \sum_{i_1,i_2}\sum_{j_1,j_2}\mathcal{K}(m+\epsilon_{j_1}+\epsilon_{i_0}|u_2)^{j_0}_{j_1}W^{i_0\,j_1}_{i_1\,j_2}(u_1+u_2)\times$$

$$\times\mathcal{K}(m+\epsilon_{j_2}+\epsilon_{i_2}|u_1)^{i_1}_{i_2}W^{j_2\,i_2}_{j_3\,i_3}(u_1-u_2)\,. \qquad (4.9)$$

Specializing m to the boundary parameter λ, i.e. $m=\lambda$, then one can easily check that the λ-independent face-type diagonal K-matrix $\mathcal{K}(\lambda|u)$ given by (4.2)–(4.3) solves (4.9). Therefore, the non-diagonal K-matrix $K^-(u)$ given by (4.1)–(4.3) satisfies the RE (2.10) associated with the R-matrix $R(u)$ given by (2.1)–(2.3).

JHEP12(2004)019

4.2 Solution of the dual RE

Similarly, one can show that the non-diagonal K-matrix $K^+(u)$ given by

$$K^+(u)_t^s = \sum_{i=1}^{n} k_i^+(u)\phi^{(s)}_{\lambda',\lambda'-\epsilon_i}(-u)\tilde{\phi}^{(t)}_{\lambda',\lambda'-\epsilon_i}(u)\,, \tag{4.10}$$

where

$$k_j^+(u) = \begin{cases} e^{-2i(j\eta)}, & 1 \le j \le l', \\ \frac{\sin(\bar{\xi}+u+\frac{n}{2}\eta)}{\sin(\bar{\xi}-u-\frac{n}{2}\eta)} e^{2i(u+\frac{n-2j}{2}\eta)}, & l'+1 \le j \le n, \end{cases} \tag{4.11}$$

satisfies the dual RE (2.14). Here $\phi_{\lambda',\lambda'-\epsilon_i}(u)$ and $\tilde{\phi}_{\lambda',\lambda'-\epsilon_i}(u)$ are respectively given by the i-th column of (3.13) (with ρ replaced by ρ') and (3.17) specializing to $m = \lambda'$, and l' is an positive integer, $1 \le l' \le n$, (which may be different from l). In addition to the discrete boundary parameter l', the solution contains $n+2$ *continuous* boundary parameters $\bar{\xi}$, ρ' and $\{\lambda'_i|i = 1, \ldots, n\}$ (which may be different from ξ, ρ and $\{\lambda_i|i = 1, \ldots, n\}$, respectively).

5. Discussions

We have constructed the intertwiner-matrix ϕ (3.13) between the two trigonometric $A^{(1)}_{n-1}$ R-matrices $R(u)$, (2.1)–(2.3), and $W(u)$, (3.10)–(3.12). From the intertwiner-matrix and its associated matrices $\bar{\phi}$ (3.16) and $\tilde{\phi}$ (3.17), we obtain a class of nondiagonal solutions $K^-(u)$, given by (4.1)–(4.3), of the RE (resp. a class of nondiagonal solutions $K^+(u)$, given by (4.10)–(4.11), of the dual RE) associated with the R-matrix $R(u)$. In addition to the discrete parameter l (resp. l'), the solution contains $n+2$ continuous boundary parameters ξ, ρ and $\{\lambda_i|i = 1, \ldots, n\}$ (resp. $\bar{\xi}$, ρ' and $\{\lambda'_i|i = 1, \ldots, n\}$).

Consider the special cases where some of the two sets of boundary parameters associated with the left and right boundaries are frozen by the restrictions,

$$\rho' = \rho\,, \qquad \lambda' + \sum_{l=1}^{N} \epsilon_{i_l} = \lambda\,, \tag{5.1}$$

where N is a positive integer, $\{i_l|l = 1, \ldots, N\}$ are positive integers such that $2 \le i_l \le n$. Then the decompositions of the K-matrices $K^{\pm}(u)$ (4.1) and (4.10) in terms of the intertwiner-matrices and diagonal face-type K-matrices should enable us to diagonalize the double-row transfer matrix of the trigonometric $A^{(1)}_{n-1}$ vertex model with non-diagonal open boundaries by means of the generalized algebraic Bethe ansatz method developed in [17]. The results will be presented elsewhere [16].

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A. The face-vertex correspondence relation

Thanks to the non-vanishing matrix elements of the R-matrices $R(u)$ (2.1)–(2.3) and $W(u)$ (3.10)–(3.12), the proof of the face-vertex correspondence relation is reduced to the proof of the relations:

$$\begin{aligned} &R_{12}(u_1-u_2)\phi_{m,m-\epsilon_i}(u_1)\otimes\phi_{m-\epsilon_i,m-2\epsilon_i}(u_2)=\\ &\qquad=W_{ii}^{ii}(u_1-u_2)\phi_{m-\epsilon_i,m-2\epsilon_i}(u_1)\otimes\phi_{m,m-\epsilon_i}(u_2)\,, \end{aligned}\tag{A.1}$$

$$\begin{aligned} &\phi^{(\alpha)}_{m,m-\epsilon_i}(u_1)\,\phi^{(\alpha)}_{m-\epsilon_i,m-\epsilon_i-\epsilon_j}(u_2)=\\ &\qquad=W_{ij}^{ij}(u_1-u_2)\phi^{(\alpha)}_{m-\epsilon_j,m-\epsilon_j-\epsilon_i}(u_1)\,\phi^{(\alpha)}_{m,m-\epsilon_j}(u_2)+\\ &\qquad\quad+W_{ij}^{ji}(u_1-u_2)\phi^{(\alpha)}_{m-\epsilon_i,m-\epsilon_i-\epsilon_j}(u_1)\,\phi^{(\alpha)}_{m,m-\epsilon_i}(u_2)\,,\qquad i\neq j\,, \end{aligned}\tag{A.2}$$

$$\begin{aligned} &R^{\alpha\beta}_{\alpha\beta}(u_1-u_2)\phi^{(\alpha)}_{m,m-\epsilon_i}(u_1)\phi^{(\beta)}_{m-\epsilon_i,m-\epsilon_i-\epsilon_j}(u_2)+\\ &+R^{\alpha\beta}_{\beta\alpha}(u_1-u_2)\phi^{(\beta)}_{m,m-\epsilon_i}(u_1)\,\phi^{(\alpha)}_{m-\epsilon_i,m-\epsilon_i-\epsilon_j}(u_2)=\\ &\qquad=W_{ij}^{ij}(u_1-u_2)\phi^{(\alpha)}_{m-\epsilon_j,m-\epsilon_j-\epsilon_i}(u_1)\,\phi^{(\beta)}_{m,m-\epsilon_j}(u_2)+\\ &\qquad\quad+W_{ij}^{ji}(u_1-u_2)\phi^{(\alpha)}_{m-\epsilon_i,m-\epsilon_i-\epsilon_j}(u_1)\,\phi^{(\beta)}_{m,m-\epsilon_i}(u_2)\,,\qquad i\neq j\,,\quad \alpha\neq\beta\,. \end{aligned}\tag{A.3}$$

With the help of (3.6)–(3.8) and the identity,

$$\sin(u)\,e^{\pm i\eta}+\sin(\eta)\,e^{\mp iu}=\sin(u+\eta),\tag{A.4}$$

eqs. (A.1) and (A.2) can be proved case by case through tedious calculations. (A.3) is further divided into three cases for $n\geq 3$:

$$\begin{aligned} &|i-j|\geq 2,\ \text{and}\ (i,j)\neq(1,n)\ \text{or}\ (n,1),\\ &|i-j|=1,\ \text{and}\ (i,j)\neq(1,n)\ \text{or}\ (n,1),\\ &(i,j)=(1,n)\ \text{or}\ (n,1), \end{aligned}$$

and one case for $n=2$:

$$(i,j)=(1,n)\ \text{or}\ (n,1)\,.$$

Using (3.6)–(3.8) and after a straightforward calculation, one can check (A.3) for all of the above cases. Therefore, we complete the proof of the face-vertex correspondence relation (3.9) which plays a key role to construct the K-matrices $K^-(u)$ (4.1) and $K^+(u)$ (4.10).

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